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PROPAGATION OF NONLINEAR WAVES IN AN INHOMOGENEOUS GAS–LIQUID MEDIUM. DERIVATION OF WAVE EQUATIONS IN THE KORTEWEG–DE VRIES APPROXIMATION

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Equations describing the propagation of waves of small but finite amplitude in a liquid with gas bubbles are derived. The bubble distribution density is a continuous function of bubble size and spatial coordinates. It is found that, for a uniform bubble distribution, the obtained equations become the Korteweg–de Vries, Kadomtsev–Petviashvili and Khokhlov–Zabolotskaya equations.

Key words: *bubble liquid, inhomogeneous medium, continuous propagation, wave equation.*

Introduction. Experimental and numerical studies of wave processes in polyphase media are considerably complicated, primarily due to the large number of characteristic parameters. Exact and approximate solutions containing problem parameters in explicit form can be used in both an analysis of experimental data and as tests in numerical calculations. A review of theoretical and experimental work on the wave mechanics of multiphase media and a bibliography of research are given in [1, 2]. The initial system of equations of motion for a liquid containing gas bubbles studied in present paper is the Iordanskii one-velocity model [3]. The validity of the replacement of the energy conservation equation by the Rayleigh equation for describing the pulsations of one bubble is shown in [4]. The method of accounting for the bubble distribution density as a continuous function of the bubble size is described in [5, 6]. In the present work, nonlinear wave equations taking into account the dependence of the bubble size distribution density on spatial coordinates are derived. Equations for one-dimensional and slightly non-one-dimensional nonlinear waves are obtained using the Korteweg–de Vries approximation.

1. Formulation of the Problem. The initial system of equations for the motion of the liquid containing gas bubbles is written as

$$\frac{\partial}{\partial t}\,[(1-\varphi)\rho] + \frac{\partial}{\partial x_k}\,[(1-\varphi)\rho u_k] = 0; \tag{1}$$

$$\frac{\partial}{\partial t}\,[(1-\varphi)\rho u_i] + \frac{\partial}{\partial x_k}\,[p\delta_{ik} + (1-\varphi)\rho u_i u_k] = 0; \tag{2}$$

$$\frac{\partial}{\partial t}\,N + \frac{\partial}{\partial x_k}\,(N u_k) = 0; \tag{3}$$

$$R\,\frac{d^2R}{dt^2} + \frac{3}{2}\Big(\frac{dR}{dt}\Big)^2 + \frac{4\nu}{R}\,\frac{dR}{dt} = \frac{1}{\rho_*}\Big[\Big(p_* + \frac{2\sigma}{R_*}\Big)\Big(\frac{R_*}{R}\Big)^{3\gamma} - \frac{2\sigma}{R} - p\Big],$$
$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_k\,\frac{\partial}{\partial x_k}; \tag{4}$$

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$$\varphi = \frac{4\pi}{3} \int\limits_0^\infty N(R_*, t, x, y, z)\, dR_*; \qquad (5)$$

$$\rho = \rho_* \Big(1 + n\, \frac{p - p_*}{\rho_* c_{1*}^2}\Big)^{1/n}. \qquad (6)$$

Here t is time, x, y, and z are spatial coordinates, $\rho(t, x, y, z)$ is the liquid density, $p(t, x, y, z)$ is the pressure, $\boldsymbol{u}(t, x, y, z)$ is the velocity, c_{1*} is the sound velocity in the liquid, σ is the surface tension, the value of the parameter n depends on the properties of the liquid (for water, $n \simeq 7.5$), $R(R_*, t, x, y, z)$ is the current radius of a bubble, R_* unperturbed radius of bubbles of the R_*-fraction, $N(R_*, t, x, y, z)$ is the bubble radius distribution density, $N(R_*, t, x, y, z)\, dR_*$ is the number of bubbles of the R_*-fractions in unit volume of the mixture, ν is the kinematic viscosity of the liquid, φ is the volumetric concentration of the gas, and γ is the adiabatic exponent for the bubble gas. The equations of conservation of the number and pulsations of bubbles are related by functions dependent on the additional variable R_*, but in the remaining equations, these functions are under the integral with respect to this variable. (We note that the integration involves no difficulties.) The initial system (1)–(6) is written in a linear approximation in φ, i.e., the parameter φ is assumed to be small. The degree of smallness of φ should provide not only the linearity of the equations of motions along φ but also the validity of the Rayleigh equation (4) describing the radial pulsations of a bubble of unperturbed radius R_* under the action of the pressure $p(t, x, y, z)$. This is the case if the following relation holds:

$$R \ll d \sim N_0^{-1/3}, \qquad N_0 = \int\limits_0^\infty N(R_*, t, x, y, z)\, dR_*.$$

Here d is the average distance between bubbles and N_0 is the total number of bubbles in unit volume of the mixture. For values of R_* smaller than the minimum and larger than the maximum radii of the bubbles present in the mixture, the function $N(R_*, t, x, y, z)$ should satisfy the condition of rapid decrease. Because φ is small, the mass of the bubble gas can be ignored; therefore, the density of the gas–liquid mixture is equal to

$$\rho_0 = \rho(1 - \varphi).$$

In addition, the initial system of equations is obtained for the case of freezing-in of the bubbles, i.e., ignoring their relative motion. To derive an approximate equation describing the propagation of weakly nonlinear waves, one should introduce some small parameter $\varepsilon \ll 1$ in explicit form in the initial system of equations. This parameter can be, for example, the ratio of the maximum velocity in the wave to the equilibrium sound velocity, the ratio of the maximum pressure pulsation to the unperturbed pressure, etc. The relations between the orders of magnitude of unknown functions and their derivatives with respect to independent variables can be found from any exact solution of the initial simplified system of equations using general considerations concerning the orders of magnitude of the nonlinear and dispersion terms of the required approximate equation.

Let us consider the solution of system (1)–(6) for the functions u, p, N, and R in the form of a plane stationary wave propagating at constant velocity V in an ideal incompressible liquid with gas bubbles of the same size

$$F = F(x - Vt), \qquad (7)$$

which satisfies the conditions

$$|x - Vt| \to \infty\colon \quad u \to 0, \quad p \to p_*, \quad N \to N_*, \quad R \to R_*. \qquad (8)$$

Substituting relations (7) and (8) into (1)–(3) and integrating the resulting equations, we obtain

$$-V(1 - \varphi) + u(1 - \varphi) = -V(1 - \varphi_*); \qquad (9)$$

$$-V(1 - \varphi)u + (1 - \varphi)u^2 + p/\rho = p_*/\rho_*; \qquad (10)$$

$$-VN + uN = -VN_*. \qquad (11)$$

Setting $u \sim \varepsilon u'$, from (9)–(11) we have

$$p - p_* \sim \varepsilon p', \quad N - N_* \sim \varepsilon N', \quad R - R_* \sim \varepsilon R', \quad V - c_* \sim \varepsilon V'. \tag{12}$$

Expression (12) can be written as

$$u = \varepsilon u', \qquad p = p_* + \varepsilon p', \qquad N = N_* + \varepsilon N', \qquad R = R_* + \varepsilon R', \qquad V = c_* + \varepsilon V', \tag{13}$$

where c_* is the sound velocity. Substituting relations (7) and (13) into (4) and performing simple calculations, we obtain the following estimate for the magnitude of the derivatives of the required functions:

$$\frac{\partial}{\partial t} \sim \frac{\partial}{\partial x} \sim \varepsilon^{1/2}. \tag{14}$$

From the last relation in (13), it follows that, in a coordinate system moving at the sound velocity c_*, the time derivative has a magnitude of order $\varepsilon^{3/2}$. Thus, for the chosen orders (13) and (14), we obtain the equation

$$u_\tau + u u_\eta + u_{\eta\eta\eta} = 0,$$

where $\eta = x - c_* t$, $\tau = \varepsilon^{3/2} t$, and all terms are of the same order of smallness $\varepsilon^{5/2}$. We assume that the presence of small viscosity, compressibility, non-one-dimensionality of waves, etc. in the initial system of equations has little effect on the chosen order of magnitude.

The unperturbed bubble size distribution density is assumed to depend weakly on the spatial coordinates

$$N_*(R_*, x, y, z) = N_*(R_*, \varepsilon^{3/2} x) + \varepsilon n_*(R_*, \varepsilon^{3/2} x, \varepsilon y, \varepsilon z). \tag{15}$$

From this it follows that the characteristic length of nonuniformity of the bubble distribution along the coordinates is much greater than the wavelength of the perturbation considered.

2. Derivation of Approximate Equations. For the adopted orders of magnitude of (13) and (14), the required functions can be written as

$$\begin{gathered} p = p_*(1 + \varepsilon p'), \qquad u = \varepsilon c_* u', \qquad v = \varepsilon^{3/2} c_* v', \\ w = \varepsilon^{3/2} c_* w', \qquad N = N_*(1 + \varepsilon N'), \qquad R = R_*(1 + \varepsilon R'), \end{gathered} \tag{16}$$

where u', v', and w' are the dimensionless velocity components along the coordinate axes, N_* is given by formula (15). The dimensionless independent coordinates can be represented as

$$x = \varepsilon^{-1/2} L x', \quad y = \varepsilon^{-1} L y', \quad z = \varepsilon^{-1} L z, \quad t = \varepsilon^{-1/2} (L/c_*) t', \quad \nu = \varepsilon^{1/2} \nu_*; \tag{17}$$

$$\eta = \int\limits_{x_0'}^{x'} \frac{d\tau}{c(\tau)} - t', \qquad \tau = \varepsilon x'. \tag{18}$$

The dimensionless functions p', u', v', w', N', and R' depend on the dimensionless variables τ, η, y', and z'. The expression for η in (18) implies transformation to a coordinate system moving in the positive x direction at variable velocity $c_* c(\tau)$, which is determined in the derivation of approximate equations. In view of (15) and the expressions for η in (18), the replacement of $\tau = \varepsilon t'$ by $\tau = \varepsilon x'$ seems logical. The constant L in (17) is a characteristic dimension of the perturbation considered, for example, the effective pulse width or the periodic signal wavelength. As noted above, the constant L should be much larger than the average distance between bubbles in the liquid.

To simplify the further calculations, we introduce the new notation. Equation (6) is written as

$$\rho = \rho_*[1 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + O(\varepsilon^3)], \qquad \rho_1 = \frac{p}{n}, \qquad \rho_2 = -\frac{n-1}{2} \frac{p^2}{n^2} \tag{19}$$

($c_{1*}^2 = n p_*/\rho_*$). Equation (5) is represented as

$$\varphi = \varphi_* + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + O(\varepsilon^3), \tag{20}$$

where

$$\varphi_*(\tau) = \frac{4\pi}{3} \int\limits_0^\infty N_*(R_*, \tau) R_*^3 \, dR_*; \tag{21}$$

$$\varphi_1(\tau,y,z)=\frac{4\pi}{3}\int\limits_0^\infty N_*(R_*,\tau)R_*^3[N(\tau,\eta,y,z)+3R(\tau,\eta,y,z)]\,dR_*+\varphi_{**}(\tau,y,z); \tag{22}$$

$$\varphi_2(\tau,y,z)=\frac{4\pi}{3}\int\limits_0^\infty N_*(R_*,\tau)[3N(\tau,\eta,y,z)R(\tau,\eta,y,z)+3R^2(\tau,\eta,y,z)]\,dR_*$$

$$+\frac{4\pi}{3}\int\limits_0^\infty n_*(R_*,\tau,y,z)R_*^3[N(\tau,\eta,y,z)+3R(\tau,\eta,y,z)]\,dR_*; \tag{23}$$

$$\varphi_{**}(\tau,y,z)=\frac{4\pi}{3}\int\limits_0^\infty n_*(R_*,\tau,y,z)R_*^3\,dR_*. \tag{24}$$

Using relations (21)–(24), we obtain

$$(1-\varphi)\rho=\rho_*[1-\varphi_*+\varepsilon a+\varepsilon^2 b+O(\varepsilon^3)], \tag{25}$$

where

$$a=(1-\varphi_*)\rho_1-\varphi_1,\qquad b=(1-\varphi_*)\rho_2-\varphi_2-\rho_2\varphi_1. \tag{26}$$

Relation (18) implies that

$$\frac{\partial}{\partial t}=-\varepsilon^{1/2}\,\frac{c_*}{L}\,\frac{\partial}{\partial\eta},\qquad \frac{\partial}{\partial x}=\varepsilon^{1/2}\,\frac{1}{c(\tau)L}\,\frac{\partial}{\partial\eta}+\tau^{3/2}\frac{1}{L}\,\frac{\partial}{\partial\tau}.$$

In view of (16)–(26), the initial system (1)–(6) becomes

$$\varepsilon^{3/2}\Big[\frac{1-\varphi_*(\tau)}{c(\tau)}\,u_\eta-a_\eta\Big]+\varepsilon^{5/2}\hat{l}_1=O(\varepsilon^{7/2}), \tag{27}$$

where $\hat{l}_1$ is an operator:

$$\hat{l}_1=[(1-\varphi_*)u]_\tau+\frac{1}{c(\tau)}\,(au)_\eta-b_\eta+(1-\varphi_*)\Big(\frac{k}{\tau}\,u+v_y+w_z\Big) \tag{28}$$

(the subscripts τ, η, y, and z denote partial derivatives with respect to the corresponding variable).

The momentum conservation equations (2) are written as

$$\varepsilon^{3/2}\Big[\frac{p_*}{c(\tau)}\,p_\eta-(1-\varphi_*)\rho_*c_*^2u_\eta\Big]+\varepsilon^{5/2}\hat{l}_2=O(\varepsilon^{7/2}); \tag{29}$$

$$\varepsilon^2[p_*p_y-(1-\varphi_*)\rho_*c_*^2v_\eta]=O(\varepsilon^3); \tag{30}$$

$$\varepsilon^2[p_*p_z-(1-\varphi_*)\rho_*c_*^2w_\eta]=O(\varepsilon^3). \tag{31}$$

Here

$$\hat{l}_2=p_*p_\tau+\frac{(1-\varphi_*)\rho_*c_*^2}{c(\tau)}\,(u^2)_\eta-\rho_*c_*^2(au)_\eta+\frac{\nu_*[1-\varphi_*(\tau)]\rho_*c_*}{L^2}\,u_{\eta\eta}. \tag{32}$$

Equation (3) has the form

$$\varepsilon^{3/2}\Big[\frac{N_*}{c(\tau)}\,u_\eta-N_*N_\eta\Big]+\varepsilon^{5/2}\hat{l}_3=O(\varepsilon^{7/2}), \tag{33}$$

where

$$\hat{l}_3=\frac{N_*}{c(\tau)}\,(Nu)_\eta+\frac{k}{\tau}\,N_*u+(N_*u)_\tau+N_*(v_y+w_z). \tag{34}$$

The bubble pulsation equation (4) is written as

$$\varepsilon(p_*p+\theta_1R)+\varepsilon^2\hat{l}_4=O(\varepsilon^3), \tag{35}$$

where

$$\hat{l}_4 = \frac{\rho_* c_*^2 R_*^2}{L^2} R_{\eta\eta} - \theta_2 R^2 - \frac{4\nu_* \rho_* c_*}{L} R_\eta; \tag{36}$$

$$\theta_1 = 3\gamma p_* + (3\gamma - 1)\frac{2\sigma}{R_*}, \qquad \theta_2 = \Big(p_* + \frac{2\sigma}{R_*}\Big)\frac{3\gamma(2\gamma+1)}{2} - \frac{2\sigma}{R_*}. \tag{37}$$

Equations (27)–(36) written in the above form can be used to solve problems of the propagation of both one-dimensional and non-one-dimensional waves. For the one-dimensional case, in these equations it is necessary to set

$$v = w = 0, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0, \qquad n_* = 0.$$

In this case, for plane, cylindrical, and spherical waves $k = 0$, 1, and 2, respectively. In the case of non-one-dimensional waves,

$$k = 0, \quad v, \quad w, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad n_* \neq 0.$$

In the above form of the initial system of equations, the small parameter ε is present in explicit form in the equations. This allows the unknown functions u, v, w, p, N, and R to be sought by their formal expansion in series in powers of the small parameter ε:

$$F = \sum_{k=0}^{\infty} F_k(\tau, \eta, y, z)\varepsilon^k.$$

In the zero approximation, the expressions for the smallest power of ε in Eqs. (27), (29), (33), and (35) should be set equal to zero:

$$\frac{1-\varphi_*(\tau)}{c(\tau)} u_{0\eta} - [1-\varphi_*(\tau)]\frac{p_*}{\rho_* c_{1*}^2} p_{0\eta} + \frac{4\pi}{3}\int_0^\infty N_*(R_*, \tau) R_*^3 (N_{9\eta} + 3R_{0\eta})\, dR_* = 0; \tag{38}$$

$$\frac{p_*}{c(\tau)} p_{0\eta} - [1-\varphi_*(\tau)]\rho_* c_*^2 u_{0\eta} = 0; \tag{39}$$

$$\frac{1}{c(\tau)} u_{0\eta} - N_{0\eta} = 0; \tag{40}$$

$$p_* p_0 + \theta_1 R_0 = 0. \tag{41}$$

From Eqs. (39)–(41), we have

$$N_0 = \frac{1}{c(\tau)} u_0, \quad p_0 = \frac{[1-\varphi_*(\tau)]\rho_* c_*^2 c(\tau)}{p_*} u_0, \quad R_0 = -\frac{[1-\varphi_*(\tau)]\rho_* c_*^2 c(\tau)}{\theta_1} u_0. \tag{42}$$

Substitution of formulas (42) into (38) yields

$$\Big\{\frac{1}{c(\tau)} - \frac{[1-\varphi_*(\tau)]^2 \rho_* c_*^2 c(\tau)}{\rho_* c_{1*}^2} - 4\pi[1-\varphi_*(\tau)]\rho_* c_*^2 c(\tau)\int_0^\infty \frac{N_*(R_*, \tau) R_*^3\, dR_*}{\theta_1}\Big\} u_0 = 0. \tag{43}$$

From relation (6), it follows that $c_{1*}^2 = np_*/\rho_*$. Equation (43) implies that, for the function u_0 to have a nonzero solution, the expression in braces should vanish. The same result can be obtained in a different way. Integrating Eqs. (38)–(40) and supplementing them by Eq. (41), we obtain a linear algebraic system of equations for the four unknown functions u_0, p_0, N_0, and R_0. For this homogeneous system to have a nontrivial solution, it is necessary that its determinant vanish, which leads to the already obtained result. The condition of vanishing of the expression in braces in (43) is given by

$$[1-\varphi_*(\tau)]\rho_* c_*^2 c^2(\tau) = np_* \theta_0(\tau), \tag{44}$$

where

$$\theta_0 = \frac{1}{1 - \varphi_*(\tau) + np_* I_1}; \tag{45}$$

$$I_1 = 4\pi \int\limits_0^\infty \frac{N_*(R_*, \tau) R_*^3 \, dR_*}{\theta_1}, \tag{46}$$

the quantity θ_1 is defined by formula (37). In view of (45), the zero solution is represented as

$$N_0 = \frac{u_0}{c(\tau)}, \qquad p_0 = n\theta_0(\tau) \frac{u_0}{c(\tau)}, \qquad R_0 = -\frac{n\theta_0(\tau) p_*}{\theta_1} \frac{u_0}{c(\tau)}. \tag{47}$$

Formula (44) defines the velocity $c_* c(\tau)$ of motion of the coordinate system chosen in relation (16) (previously, this velocity was undefined). The function u_0 in the zero approximation remains arbitrary. To obtain an equation for the function u_0, it is necessary to consider the first approximation. Equations (27), (29), (30), (31), (33), and (35) in the first approximation have the form

$$\frac{1 - \varphi_*(\tau)}{c(\tau)} u_{1\eta} - [1 - \varphi_*(\tau)] \frac{p_*}{\rho_* c_{1*}^2} p_{1\eta} + \frac{4\pi}{3} \int\limits_0^\infty N_*(R_*, \tau) R_*^3 (N_{1\eta} + 3R_{1\eta}) \, dR_* + \hat{l}_1(u_0) = 0; \tag{48}$$

$$\frac{p_*}{c(\tau)} p_{1\eta} - [1 - \varphi_*(\tau)] \rho_* c_*^2 u_{1\eta} + \hat{l}_2(u_0) = 0,$$

$$\frac{N_*(R_*, \tau)}{c(\tau)} u_{1\eta} - N_*(R_*, \tau) N_{1\eta} + \hat{l}_3(u_0) = 0, \tag{49}$$

$$p_* p_1 + \theta_1 R_1 + \hat{l}_4(u_0) = 0;$$

$$p_* p_{0y} - [1 - \varphi_*(\tau)] \rho_* c_*^2 v_{0\eta} = 0, \qquad p_* p_{0z} - [1 - \varphi_*(\tau)] \rho_* c_*^2 w_{0\eta} = 0. \tag{50}$$

From Eqs. (49), we obtain

$$p_{1\eta} = \frac{[1 - \varphi_*(\tau)] \rho_* c_*^2 c(\tau)}{p_*} u_{1\eta} - \frac{c(\tau)}{p_*} \hat{l}_2(u_0),$$

$$N_{1\eta} = \frac{1}{c(\tau)} u_{1\eta} + \frac{1}{N_*} \hat{l}_3(u_0), \tag{51}$$

$$R_{1\eta} = \frac{[1 - \varphi_*(\tau)] \rho_* c_*^2 c(\tau)}{\theta_1} u_{1\eta} + \frac{c(\tau)}{\theta_1} \hat{l}_2(u_0) - \frac{1}{\theta_1} [\hat{l}_4]_\eta$$

[$\hat{l}_i(u_0)$ are defined by formulas (28), (32), (34), and (36) in view of (47)].

Substitution of relations (51) into Eq. (48) yields

$$\Big\{ \frac{1}{c(\tau)} - \frac{[1 - \varphi_*(\tau)]^2 \rho_* c_*^2 c(\tau)}{\rho_* c_{1*}^2} - 4\pi [1 - \varphi_*(\tau)] \rho_* c_*^2 c(\tau) \int\limits_0^\infty \frac{N_*(R_*, \tau) R_*^3 \, dR_*}{\theta_1} \Big\} u_{1\eta}$$

$$+ \hat{l}_1(u_0) + \frac{\hat{l}_2(u_0)}{1 - \varphi_*(\tau)} \rho_* c_*^2 c(\tau) + \frac{4\pi}{3} \int\limits_0^\infty R_*^3 \hat{l}_3(u_0) \, dR_* - 4\pi \int\limits_0^\infty \frac{N_*(R_*, \tau) R_*^3 [\hat{l}_4(u_0)]_\eta \, dR_*}{\theta_1} = 0. \tag{52}$$

From Eqs. (42) and (43), it follows that the expression in braces in (52) is equal to zero, and the remaining part of Eq. (52) and Eq. (50) is a system of equations for the quantities u_0, v_0, and w_0. In view of relations (27), (29), (33), (35), and (43)–(50) this system of equations is written as

$$u_{0\tau} - c_1(\tau, y, z) u_{0\eta} + \alpha(\tau) u_0 u_{0\eta} + \beta(\tau) u_{0\eta\eta\eta} - \mu(\tau) u_{0\eta\eta} + [k/(2\tau) + \delta(\tau)] u_0 + (v_{0y} + w_{0z})/2 = 0; \tag{53}$$

$$v_{0\eta} = c(\tau) u_{0y}, \qquad w_{0\eta} = c(\tau) u_{0z}, \tag{54}$$

where

$$
\begin{gathered}
c_1(\tau,y,z)=\frac{1}{2c(\tau)}\Big(np_*\theta_0(\tau)I_{1*}(\tau,y,z)-\frac{\varphi_{**}(\tau,y,z)}{1-\varphi_*(\tau)}\Big),\\
\alpha(\tau)=\frac{\theta_0^2(\tau)}{c^2(\tau)}\Big(\frac{n+1}{2}\,[1-\varphi_*(\tau)]+n^2p_*^2(I_2+I_3)\Big),\\
\beta(\tau)=\frac{n^2p_*^2\theta_0^2(\tau)}{2c^3(\tau)[1-\varphi_*(\tau)]L^2}\,I_4,\qquad \mu(\tau)=\frac{2\nu_*n^2p_*^2\theta_0^2(\tau)}{c^3(\tau)[1-\varphi_{*2}(\tau)]Lc_*}\,I_2,\\
\delta(\tau)=\frac{1}{2[1-\varphi_*(\tau)]c(\tau)}\,\{[1-\varphi_*(\tau)]c(\tau)\}_\tau;\\
I_{1*}(\tau,y,z)=4\pi\int\limits_0^\infty\frac{n_*(R_*,\tau,y,z)R_*^3\,dR_*}{\theta_1},\qquad I_2(\tau)=4\pi\int\limits_0^\infty\frac{N_*(R_*,\tau)R_*^3\,dR_*}{\theta_1^2},\\
I_3(\tau)=4\pi\int\limits_0^\infty\frac{N_*(R_*,\tau)R_*^3\theta_2\,dR_*}{\theta_1^3},\qquad I_4(\tau)=4\pi\int\limits_0^\infty\frac{N_*(R_*,\tau)R_*^5\,dR_*}{\theta_1^2}.
\end{gathered}
\tag{55}
$$

For one-dimensional waves, $c_1=v_0=w_0=0$; for plane, cylindrical, and spherical waves, $k=1$, 2, and 3, respectively; for non-one-dimensional waves, $k=0$.

System (53), (54) is the required approximate system for describing the propagation of nonlinear waves in an inhomogeneous gas–liquid medium. As noted above, for one-dimensional nonlinear waves in an inhomogeneous gas–liquid medium, system (53), (54) reduces to one equation for u (the subscripts 0 is omitted):

$$u_\tau-c_1(\tau)u_\eta+\alpha(\tau)uu_\eta+\beta(\tau)u_{\eta\eta\eta}-\mu(\tau)u_{\eta\eta}+[k/(2\tau)+\delta(\tau)]u=0 \tag{56}$$

[$c_1(\tau)$, $\alpha(\tau)$, $\beta(\tau)$, $\mu(\tau)$, and $\delta(\tau)$ are defined in (55); $k=0$, 1, and 2 for plane, cylindrical, and spherical waves, respectively]. For one-dimensional waves in the case of variable coefficients, the momentum and energy conservation laws for $\mu(\tau)=0$ and $u(\tau,\eta)\to 0$ as $|\eta|\to\infty$ have the form

$$[1-\varphi_*(\tau)]\tau^kc(\tau)\int\limits_{-\infty}^{\infty}u(\tau,\eta)\,d\eta=C_1+O(\varepsilon); \tag{57}$$

$$[1-\varphi_*(\tau)]\tau^kc(\tau)\int\limits_{-\infty}^{\infty}u^2(\tau,\eta)\,d\eta=C_2+O(\varepsilon). \tag{58}$$

It is easy to show that, for zero viscosity, the energy conservation law (58) is satisfied for any exact solution of Eq. (56). For the momentum conservation law (57) to be satisfied in the case where $\delta(\tau)\neq 0$, it is sufficient that the condition $C_1=0$ be satisfied at the initial time, since Eq. (56) implies that the condition of vanishing momentum is valid for any subsequent time. If $N_*(R_*,\tau)=N_*(R_*)$, i.e., the bubble size distribution density is identical at all points of space, the coefficients α, β, and μ are constants, $\delta=0$, and Eq. (56) becomes the classical Korteweg–de Vries–Burgers equation

$$u_\tau+\alpha uu_\eta+\beta u_{\eta\eta\eta}-\mu u_{\eta\eta}+ku/(2\tau)=0.$$

For $k=0$, from relations (53) and (54) we obtain the following system of equations describing the propagation of nonlinear non-one-dimensional waves (the subscript 0 is omitted):

$$
\begin{gathered}
u_\tau+\alpha(\tau)uu_\eta+\beta(\tau)u_{\eta\eta\eta}-\mu(\tau)u_{\eta\eta}+\delta(\tau)u-c_1(\tau,y,z)u_\eta+(v_y+w_z)/2=0,\\
v_\eta=c(\tau)u_y,\qquad w_\eta=c(\tau)u_z.
\end{gathered}
\tag{59}
$$

Eliminating the functions v and w from relations (59), for u we obtain the equation

$$[u_\tau+\alpha(\tau)uu_\eta+\beta(\tau)u_{\eta\eta\eta}-\mu(\tau)u_{\eta\eta}+\delta(\tau)u-c_1(\tau,y,z)u_\eta]_\eta+c(\tau)(u_{yy}+u_{zz})/2=0. \tag{60}$$

An equation similar to Eq. (60) was first obtained in [7] for surface waves in a tank of finite depth [$c(\tau) = 1$ and $\mu(\tau) = \delta(\tau) = 0$; α and β are constants]. Later, this equation (for $c_1 = 0$) became known as the Kadomtsev–Petviashvili equation. Elimination of the dispersion term from Eq. (60) yields the well-known Khokhlov–Zabolotskaya equation derived for the propagation of nonlinear sound beams in a viscous liquid. As in the case of one-dimensional waves, to Eq. (60) we add the momentum and energy conservation laws. For $\mu = 0$ and $u \to 0$ as $|\eta| \to \infty$, $|y| \to \infty$, and $|z| \to \infty$, the conservation laws have the form

$$[1 - \varphi_*(\tau)] \iiint\limits_{-\infty}^{\infty} u(\tau, \eta, y, z)\, d\eta\, dy\, dz = C_1 + O(\varepsilon),$$

$$[1 - \varphi_*(\tau)] \iiint\limits_{-\infty}^{\infty} u^2(\tau, \eta, y, z)\, d\eta\, dy\, dz = C_2 + O(\varepsilon),$$

where C_1 and C_2 are constants. The propagation of signals generated by an axisymmetric source is described by the equation

$$[u_\tau + \alpha(\tau) u u_\eta + \beta(\tau) u_{\eta\eta\eta} - \mu(\tau) u_{\eta\eta} + \delta(\tau) u - c_1(\tau, r) u_\eta]_\eta + c(\tau)(u_{rr} + u_r/r)/2 = 0.$$

In this case for $\mu = 0$ and $u \to 0$ as $|\eta| \to \infty$ and $r \to \infty$, the momentum and energy conservation laws have the form

$$[1 - \varphi_*(\tau)] \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} u(\tau, \eta, r) r\, dr\, d\eta = C_1 + O(\varepsilon),$$

$$[1 - \varphi_*(\tau)] \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} u^2(\tau, \eta, r) r\, dr\, d\eta = C_2 + O(\varepsilon).$$

3. Conclusions. The study performed leads to the following conclusions.

The orders of magnitudes of the unknown functions and degrees of stretching of the independent coordinates are chosen so that, for the smallest power of the small parameter ε, only linear integrated equations are retained. This leads to a linear homogeneous algebraic system for the unknown functions and allows the velocity of the coordinate system to be determined with any of the unknown functions kept arbitrary. It is for this reason that the terms proportional to k/τ in Eqs. (27) and (33) can be considered terms of the order of smallness $\varepsilon^{5/2}$. This allows one to obtain the nonlinear equation (53) by formal expansion of the unknown functions in series in powers of the small parameter, which leads, as a rule, to a sequence of only linear equations.

The chosen order of smallness of the terms proportional to k/τ implies that, in the case of one-dimensional cylindrical and spherical waves, Eq. (53) is valid only at large distances from the coordinate origin, i.e., for $\tau_0 \geqslant \varepsilon^{3/2}$. At smaller distances, along with the waves propagating in the positive direction, a significant contribution comes from waves of opposite direction, i.e., at such distances, it is not correct to distinguish waves of one direction.

The order of smallness $\varepsilon^{3/2}$ of $\partial/\partial\tau$ provides for the elimination of the term $\delta(\tau)u$ from the system of equations for the smallest power of ε, which allows for integration of this system. In the plane case, unlike in the cylindrical and spherical cases, there is no restriction on τ_0 since the smallness of the derivative with respect to τ is sufficient for the required smallness of the corresponding terms of Eqs. (27) and (33).

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